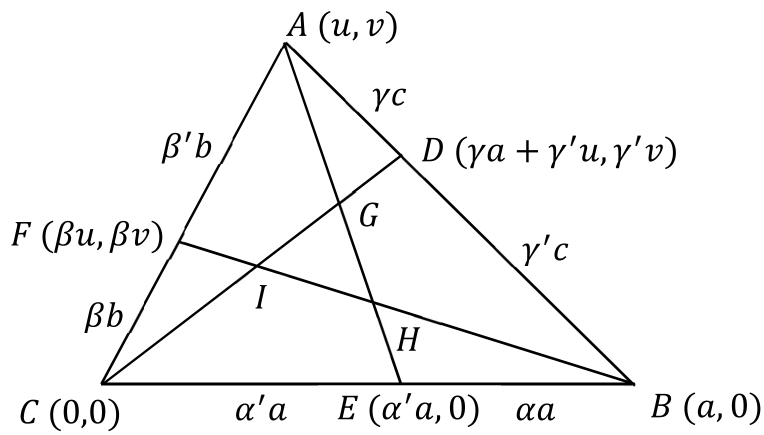
Routh's Theorem and Related Relationships

Consider a triangle *ABC* with sides a, b and c as shown below, where the labels on points give their rectangular coordinates (x, y) in brackets. The points D, E, and F divide the sides as shown, where α, β and γ have values in the range $0\,..\,1$ with $\alpha + \alpha' = \beta + \beta' = \gamma + \gamma' = 1$. Routh's theorem gives the area of triangle *GHI* in terms of the area of triangle *ABC.*



Results will also be given in Routh's form which uses the ratios of the two parts of each side $1\!:\!x, 1\!:\!y$ and $1\!:\!z$ in place of α, β and γ.

Using the equation for a line through two points, the relationships between x and y on the lines *AE* can be derived as:

$$x = \left(\frac{u - \alpha' a}{v}\right) y + \alpha' a \tag{1}$$

Similarly, for line *BF*:

$$x = \left(\frac{\beta u - a}{\beta v}\right) y + a \tag{2}$$

and line *CD*:

$$x = \left(\frac{\gamma a + \gamma' u}{\gamma' v}\right) y \tag{3}$$

The height (y_I) of point *I* above the baseline can now be determined by solving for the intersection of lines *BF* and *CD* using equations (2) and (3) to give:

$$\frac{y_I}{v} = \frac{\beta(1-\gamma)}{1 - \gamma + \beta\gamma} \tag{4}$$

And, since triangles ABC and BCI have the same base, the areas are in the same ratio.

We hence obtain:

$$\frac{\Delta BCI}{\Delta ABC} = \frac{\beta(1-\gamma)}{1-\gamma+\beta\gamma} = \frac{y}{1+y+yz} \tag{5}$$

By cyclically rotating the values α, β and γ in this equation we can obtain:

$$\frac{\Delta CAG}{\Delta ABC} = \frac{\gamma(1-\alpha)}{1-\alpha+\gamma\alpha} = \frac{z}{1+z+zx} \tag{6}$$

and

$$\frac{\Delta ABH}{\Delta ABC} = \frac{\alpha(1-\beta)}{1-\beta+\alpha\beta} = \frac{x}{1+x+xy} \tag{7}$$

Since the area of triangle ABC is the sum of the areas of the four triangles BCI, CAG, ABH and GHI, the area of the latter can now be expressed as:

$$\frac{\Delta GHI}{\Delta ABC} = 1 - \left\{\frac{\beta(1-\gamma)}{1-\gamma+\beta\gamma} + \frac{\gamma(1-\alpha)}{1-\alpha+\gamma\alpha} + \frac{\alpha(1-\beta)}{1-\beta+\alpha\beta}\right\} \tag{8}$$

which reduces to:

$$\frac{\Delta GHI}{\Delta ABC} = \frac{\{1-(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\gamma\alpha)-2\alpha\beta\gamma\}^2}{(1-\alpha+\gamma\alpha)(1-\beta+\alpha\beta)(1-\gamma+\beta\gamma)} \tag{9}$$

In Routh's form this becomes:

$$\frac{\Delta GHI}{\Delta ABC} = \frac{(xyz-1)^2}{(1+x+xz)(1+y+yx)(1+z+zy)} \tag{10}$$

The height (y_H) of point *H* above the baseline can be determined similarly by solving for the intersection of lines *AE* and *BF* with equations (1) and (2) giving:

$$\frac{y_H}{v} = \frac{\alpha\beta}{1-\beta+\alpha\beta} \tag{11}$$

Since the base of triangle *BEH* is αa we can now derive the ratio of the areas of triangles *BEH* and *ABC* as:

$$\frac{\Delta BEH}{\Delta ABC} = \frac{\alpha^2\beta}{1-\beta+\alpha\beta} = \frac{x^2 y}{(1+x)(1+x+xy)} \tag{12}$$

By cyclically rotating the values α, β and γ in this equation we can now obtain:

$$\frac{\Delta CFI}{\Delta ABC} = \frac{\beta^2\gamma}{1-\gamma+\beta\gamma} = \frac{y^2 z}{(1+y)(1+y+yz)} \tag{13}$$

and

$$\frac{\Delta ADG}{\Delta ABC} = \frac{\gamma^2\alpha}{1-\alpha+\gamma\alpha} = \frac{z^2 x}{(1+z)(1++zx)} \tag{14}$$

These and earlier area relationships now allow all seven sub areas of triangle *ABC* to be determined.